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Y. W. LEE & M. SCHETZEN

Department of Electrical Engineering and Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts, U.S.A.


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Measurement of the Wiener Kernels of a Non-linear System
by Cross-correlation†

By Y. W. Lee and M. Schetzen‡

Department of Electrical Engineering and Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts, U.S.A.

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ABSTRACT

A practical and relatively simple method of measuring the Wiener kernels of a non-linear system is presented. The method is based upon cross-correlation techniques and avoids orthogonal expansions such as those of the Wiener method of measurement. The application of this method to the experimental characterization of a non-linear system is discussed.

§ 1. Introduction

In the Wiener theory of non-linear systems (Wiener 1958) the input \( x(t) \) of a system \( A \), as shown in fig. 1, is a white Gaussian process. The output \( y(t) \) of the system is represented by the orthogonal expansion:

\[
y(t) = \sum_{n=0}^{\infty} G_n[h_n, x(t)].
\]

(1)

in which \( \{h_n\} \) is the set of Wiener kernels of the non-linear system, and \( \{G_n\} \) is a complete set of orthogonal functionals. The orthogonal property

A non-linear system with a white Gaussian noise input.

of the functionals is expressed by the fact that the time average

\[
G_n[h_n, x(t)]G_m[h_m, x(t)] = 0 \quad \text{for } m \neq n.
\]

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‡ Present address: Department of Electrical Engineering, Northeastern University, Boston, Mass., U.S.A.
For reference, we list the first four $G$-functionals:

\[ G_0[h_0, x(t)] = h_0, \]
\[ G_1[h_1, x(t)] = \int_{-\infty}^{\infty} h_1(\tau_1) x(t - \tau_1) d\tau_1, \]
\[ G_2[h_2, x(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_2(\tau_1, \tau_2) x(t - \tau_1) x(t - \tau_2) d\tau_1 d\tau_2 \]
\[ -K \int_{-\infty}^{\infty} h_2(\tau_2, \tau_2) d\tau_2, \]
\[ G_3[h_3, x(t)] = \left\{ \begin{array}{l}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_3(\tau_1, \tau_2, \tau_3) x(t - \tau_1) x(t - \tau_2) x(t - \tau_3) \\
\times d\tau_1 d\tau_2 d\tau_3 - 3K \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_3(\tau_1, \tau_2, \tau_2) x(t - \tau_1) d\tau_1 d\tau_2 \end{array} \right. \quad (2) \]

The power density spectrum of the Gaussian input, $x(t)$, is $\Phi_{xx}(\omega) = K/2\pi$ watts per radian per second so that the autocorrelation function of the input is $\phi_{xx}(\tau) = Ku(\tau)$, where $u(\tau)$ is the unit impulse function. The leading term of the $n$th-degree functional, $G_n$, is:

\[ \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n) x(t - \tau_1) \ldots x(t - \tau_n) d\tau_1 \ldots d\tau_n, \]

which is a homogeneous functional of the $n$th degree. The other terms of $G_n$ are each a homogeneous functional of degree lower than $n$ whose kernel is derived in a systematic manner from the kernel of the leading term, so that for the input $x(t)$ the functional $G_n$ is orthogonal to all functionals of degrees lower than $n$. This is seen in the expressions (2) for the first four $G$ functionals. It can be shown that the coefficients of the terms of $G_n$ are the coefficients of the terms of the Hermite polynomials:

\[ H_n(u) = u^n - C_{2^n} K u^{n-2} + 1 \cdot 3 \cdot C_4 K u^{n-4} - 1 \cdot 3 \cdot 5 \cdot C_6 K u^{n-6} + \ldots, \quad (3) \]

in which $u$ is a real variable, and $C_{2^n}$ are the binomial coefficients.

In this manner, a non-linear system is characterized by the set of Wiener kernels $\{h_n\}$. The zero-order Wiener kernel, $h_0$, is a constant; the first-order Wiener kernel, $h_1(\tau_1)$, is the linear kernel or the unit impulse response of a linear system. The second-order Wiener kernel, or the quadratic kernel, is $h_2(\tau_1, \tau_2)$, and the $n$th-order Wiener kernel is $h_n(\tau_1, \ldots, \tau_n)$. Note that the first-order Wiener kernel is not necessarily the total linear kernel of the system; similarly, the $n$th-order Wiener kernel is not necessarily the total $n$th-order kernel of the system. The determination of the kernels is a major problem in the Wiener theory. Wiener expands the kernels in terms of a set of orthogonal functions such as the Laguerre functions. Thus if $\{l_m(\tau)\}$ is the set of Laguerre functions, then

\[ h_1(\tau_1) = \sum_{m=0}^{\infty} c_m l_m(\tau_1), \]
\[ h_2(\tau_1, \tau_2) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} c_{m_1 m_2} l_{m_1}(\tau_1) l_{m_2}(\tau_2), \]
\[ \ldots \]
\[ h_n(\tau_1, \ldots, \tau_n) = \sum_{m_1=0}^{\infty} \ldots \sum_{m_n=0}^{\infty} c_{m_1 \ldots m_n} l_{m_1}(\tau_1) \ldots l_{m_n}(\tau_n). \]
The determination of the coefficients of the Laguerre expansions, which leads to the determination of the $G$-functionals, is accomplished by a system of measurements.

We wish to introduce a method of determining the Wiener kernels of a non-linear system which depends upon cross-correlation techniques and avoids orthogonal expansions such as those of eqn. (4). This method is an extension of the cross-correlation method that has been applied to linear systems (Lee 1960).

§ 2. MULTI-DIMENSIONAL-DELAY WHITE GAUSSIAN PROCESSES

We introduce a set of functionals that are formed by passing a white Gaussian noise through a system of delay circuits as shown in fig. 2. In fig. 2(a) we have a delay circuit $B$ with an adjustable delay time of $\sigma$

![Fig. 2](image)

Delay circuits: (a) one-dimensional-delay circuit, (b) two-dimensional-delay circuit, (c) three-dimensional-delay circuit.

(seconds). The input $x(t)$ is a white Gaussian process whose power density spectrum is $K/2\pi$ watts per radian per second. The output $y_1(t)$ of the delay circuit is:

$$y_1(t) = x(t - \sigma), \quad (5)$$

which can be written in the form of eqn. (3) as:

$$y_1(t) = \int_{-\infty}^{\infty} u(\tau - \sigma)x(t - \tau)\,d\tau, \quad (6)$$
in which \( u(t) \) is the unit impulse function. The integral in eqn. (6) is a functional of first degree. We shall call \( y_1(t) \) a one-dimensional-white Gaussian process.

In similar manner we form a white Gaussian process with a two-dimensional delay as shown in fig. 2 (b). Applying \( x(t) \) to the delay circuits \( B_1 \) and \( B_2 \) whose adjustable delay times are \( \sigma_1 \) and \( \sigma_2 \) and multiplying the outputs of \( B_1 \) and \( B_2 \) to form the output \( y_2(t) \) of the system, we have:

\[
y_2(t) = x(t - \sigma_1)x(t - \sigma_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(\tau_1 - \sigma_1)u(\tau_2 - \sigma_2)x(t - \tau_1) \times x(t - \tau_2) d\tau_1 d\tau_2. \tag{7}
\]

This expression is a homogeneous functional of second degree. We shall refer to \( y_2(t) \) as a two-dimensional-delay white Gaussian process.

In fig. 2 (c) we have \( t(x) \) applied to three delay circuits \( B_1, B_2, \) and \( B_3 \) whose adjustable delay times are \( \sigma_1, \sigma_2, \) and \( \sigma_3, \) and the outputs of the circuits are multiplied so that the product, which is the output of the whole system, is:

\[
y_3(t) = x(t - \sigma_1)x(t - \sigma_2)x(t - \sigma_3)
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(\tau_1 - \sigma_1)u(\tau_2 - \sigma_2)u(\tau_3 - \sigma_3)x(t - \tau_1)x(t - \tau_2)x(t - \tau_3) \times x(t - \tau_3) d\tau_1 d\tau_2 d\tau_3. \tag{8}
\]

This is a three-dimensional-delay white Gaussian process, and a homogeneous functional of third degree. Obviously, the \( n \)-dimensional-delay white Gaussian process is:

\[
y_n(t) = x(t - \sigma_1) \ldots x(t - \sigma_n)
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} u(\tau_1 - \sigma_1) \ldots u(\tau_n - \sigma_n)x(t - \tau_1) \ldots \times x(t - \tau_n) d\tau_1 \ldots d\tau_n. \tag{9}
\]

The use of these functionals in the measurement of isolated kernels has been discussed by George (1959). In the general case of a non-linear system which has more than one kernel, he resorted to a Taylor series expansion. The method that we present here does not depend upon expansions of the kernels in any form.

§ 3. Determination of the Zero- and First-Order Wiener Kernels

Consider that the non-linear system \( A \) in fig. 1 is to be characterized; that is, the set of Wiener kernels \( \{h_n\} \) of \( A \) are to be determined. The zero-order kernel, \( h_0, \) is just \( y(t) \), the average value of the output \( y(t) \) for the input \( x(t) \). The first-order Wiener kernel is measured, as indicated in fig. 3, by applying \( x(t) \) to \( A \) and the delay circuit \( B \) of fig. 2 (a), multiplying their outputs \( y(t) \) and \( y_1(t) \), and averaging the product. The average is:

\[
\overline{y(t)y_1(t)} = \left( \sum_{n=0}^{\infty} \mathcal{G}_e [h_n, x(t)] \right) x(t - \sigma). \tag{10}
\]
Since $x(t-\sigma)$ is a functional of the first degree, the functionals $G_n$, for $n > 1$, are orthogonal to $x(t-\sigma)$. Furthermore, for $n = 0$, we have

$$G_n[h_0, x(t)]x(t-\sigma) = \overline{h_0} x(t-\sigma) = 0.$$ 

Hence, with $G_1$ as given in eqn. (2), we have:

$$y(t) y_1(t) = \int_{-\infty}^{\infty} h_1(\tau_1) x(t-\tau_1) d\tau_1 = \int_{-\infty}^{\infty} h_1(\tau_1) x(t-\tau_1) x(t-\sigma) d\tau_1$$

$$= \int_{-\infty}^{\infty} h_1(\tau_1) Ku(\sigma-\tau_1) d\tau_1 = K h_1(\sigma).$$

Therefore, by applying a white Gaussian process to the unknown non-linear system $A$ and to the one-dimensional-delay circuit $B$, and then averaging the product of their outputs for various values of the delay time $\sigma$, we obtain the first-order Wiener kernel of the non-linear system:

$$h_1(\sigma) = \frac{1}{K} \overline{y(t)y_1(t)}. \quad (12)$$

### § 4. Determination of the Second-order Wiener Kernel

To measure the second-order Wiener kernel, we connect the system of fig. 2 (b) to the unknown non-linear system $A$ in the manner shown in fig. 4. The average of the product of the outputs of the unknown non-linear system and the two-dimensional-delay circuit is:

$$\overline{y(t)y_2(t)} = \left\{ \sum_{n=0}^{\infty} G_n[h_n, x(t)] \right\} x(t-\sigma_1)x(t-\sigma_2). \quad (13)$$

We note that the functionals $G_n$ for $n > 2$ are orthogonal to $x(t-\sigma_1)x(t-\sigma_2)$, which is a homogeneous functional of second degree. For $n = 0$, the average involving $G_0$ is:

$$G_0[h_0, x(t)]x(t-\sigma_1)x(t-\sigma_2) = \overline{h_0 x(t-\sigma_1)x(t-\sigma_2)}$$

$$= \overline{h_0 Ku(\sigma_1-\sigma_2)}. \quad (14)$$
Measurement of the second-order Wiener kernel of a non-linear system.

The average for $n = 1$ is:

$$G_1[h_1(t), x(t)] x(t - \sigma_1) x(t - \sigma_2) = \left[ \int_{-\infty}^{\infty} h_1(\tau_1) x(t - \tau_1) \, d\tau_1 \right] x(t - \sigma_1) x(t - \sigma_2)$$

$$= \int_{-\infty}^{\infty} h_1(\tau_1) x(t - \tau_1) x(t - \sigma_1) x(t - \sigma_2) \, d\tau_1 = 0,$$

(15)
since the average of the product of an odd number of zero-mean Gaussian variables is zero (Schetzen 1961). Finally, the average for $n = 2$ involving $G_2$ as given in eqn. (2), is:

$$G_2[h_2(t), x(t)] x(t - \sigma_1) x(t - \sigma_2)$$

$$= \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_2(\tau_1, \tau_2) x(t - \tau_1) x(t - \tau_2) \, d\tau_1 \, d\tau_2 - K^2 \int_{-\infty}^{\infty} h_2(\tau_2, \tau_2) \, d\tau_2 \right] x(t - \sigma_1) x(t - \sigma_2)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_2(\tau_1, \tau_2) x(t - \tau_1) x(t - \tau_2) x(t - \sigma_1) x(t - \sigma_2) \, d\tau_1 \, d\tau_2$$

$$- K^2 u(\sigma_1 - \sigma_2) \int_{-\infty}^{\infty} h_2(\tau_2, \tau_2) \, d\tau_2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_2(\tau_1, \tau_2) K^2 [u(\tau_1 - \tau_2) u(\sigma_1 - \sigma_2) + u(\tau_1 - \sigma_1) u(\tau_2 - \sigma_2)]$$

$$+ u(\tau_1 - \sigma_2) u(\tau_2 - \sigma_1) \, d\tau_1 \, d\tau_2 - K^2 u(\sigma_1 - \sigma_2) \int_{-\infty}^{\infty} h_2(\tau_2, \tau_2) \, d\tau_2$$

$$= K^2 \left[ u(\sigma_1 - \sigma_2) \int_{-\infty}^{\infty} h_2(\tau_1, \tau_1) \, d\tau_1 + h_2(\sigma_1, \sigma_1) + h_2(\sigma_2, \sigma_2) \right]$$

$$- u(\sigma_1 - \sigma_2) \int_{-\infty}^{\infty} h_2(\tau_2, \tau_2) \, d\tau_2 \right] = 2K^2 h_2(\sigma_1, \sigma_2).$$

(16)
The kernels in eqn. (1) are symmetrical in the variables $\tau_1, \ldots, \tau_n$ so that for the second-order kernel we have $h_2(\tau_1, \tau_2) = h_2(\tau_2, \tau_1)$. Therefore, our final result for eqn. (13) is:

$$y(t)y_2(t) = y(t)x(t - \tau_1)x(t - \tau_2) = 2K^2h_2(\sigma_1, \sigma_2) + h_0Ku(\sigma_1 - \sigma_2).$$ (17)

The first term on the right-hand side of this equation involves the third-order Wiener kernel of the non-linear system that we wish to determine. The second term on the same side of the equation gives rise to an impulse when $\sigma_1 = \sigma_2$. But, when $\sigma_1 \neq \sigma_2$, the term has zero value. Although theoretically the method does not yield the values of the second-order kernel at $\sigma_1 = \sigma_2$, we should have no difficulty in practical application of the method because we can come as close as we please to these points. A procedure by which this restriction is removed, is given in §7. The result in eqn. (17) means that if we apply $x(t)$ to the unknown non-linear system and to the two-dimensional-delay circuit and then take the average of the product of their outputs for various values of the delay times $\sigma_1$ and $\sigma_2$, we shall have the second-order Wiener kernel of the unknown non-linear system given by:

$$h_2(\sigma_1, \sigma_2) = \frac{1}{2K^2} \overline{y(t)y_2(t)}; \quad \text{for } \sigma_1 \neq \sigma_2.$$ (18)

§ 5. Determination of the Third-order Wiener Kernel

In a manner similar to the measurement of the first- and second-order kernels we measure the third-order Wiener kernel of a non-linear system as indicated in fig. 5. The average of the product of the output of the

![Fig. 5](image_url)
unknown non-linear system and the output of the three-dimensional

delay circuit as a function of the delay times \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) is:

\[
\bar{y}(t) = \left\{ \sum_{n=0}^{\infty} G_n[h_n, x(t)] \right\} x(t-\sigma_1) x(t-\sigma_2) x(t-\sigma_3).
\]  

(19)

Since \( x(t-\sigma_1)x(t-\sigma_2)x(t-\sigma_3) \) is a homogeneous functional of the third
degree, it is orthogonal to \( G_n \) for \( n > 3 \). For \( n = 3 \), with \( G_3 \) as given in
eqn. (3), we have:

\[
G_3[h_3, x(t)] x(t-\sigma_1)x(t-\sigma_2)x(t-\sigma_3)
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_3(\tau_1, \tau_2, \tau_3) x(t-\tau_1)x(t-\tau_2)x(t-\tau_3) d\tau_1 d\tau_2 d\tau_3
\]

\[
-3K \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_3(\tau_1, \tau_2, \tau_3) x(t-\sigma_1)x(t-\sigma_2)x(t-\sigma_3) d\tau_1 d\tau_2 d\tau_3
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_3(\tau_1, \tau_2, \tau_3)
\times x(t-\tau_1)x(t-\tau_2)x(t-\tau_3) x(t-\sigma_1)x(t-\sigma_2)x(t-\sigma_3) d\tau_1 d\tau_2 d\tau_3
\]

\[
-3K \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_3(\tau_1, \tau_2, \tau_3) x(t-\tau_1)x(t-\tau_2)x(t-\tau_3) x(t-\sigma_1)x(t-\sigma_2)x(t-\sigma_3) d\tau_1 d\tau_2 d\tau_3.
\]  

(20)

The triple integral of eqn. (18) can be shown to be equal to:

\[
K^3 \left[ 6h_3(\sigma_1, \sigma_2, \sigma_3) + 3u(\sigma_2-\sigma_3) \int_{-\infty}^{\infty} h_3(\tau_1, \tau_1, \sigma_1) d\tau_1 + 3u(\sigma_1-\sigma_2) \right]
\times \int_{-\infty}^{\infty} h_3(\tau_3, \tau_3, \sigma_3) d\tau_3 + 3u(\sigma_3-\sigma_1) \int_{-\infty}^{\infty} h_3(\tau_2, \tau_2, \sigma_2) d\tau_2
\]

(21)

and the last term of eqn. (18) can be shown to be equal to:

\[
-3K^3 \left[ u(\sigma_1-\sigma_2) \int_{-\infty}^{\infty} h_3(\tau_1, \tau_1, \sigma_3) d\tau_1 + u(\sigma_2-\sigma_1) \int_{-\infty}^{\infty} h_3(\tau_1, \tau_1, \sigma_2) d\tau_1 \right]
\]

\[
+ u(\delta_2-\sigma_3) \int_{-\infty}^{\infty} h_3(\tau_1, \tau_1, \sigma_1) d\tau_1
\]

(22)

Hence eqn. (20) reduces to:

\[
\bar{G}_3[h_3, x(t)] x(t-\sigma_1)x(t-\sigma_2)x(t-\sigma_3) = 6K^3 h_3(\sigma_1, \sigma_2, \sigma_3).
\]  

(23)

To complete the evaluation of eqn. (19), we need to consider the average of
the product of the three-dimensional-delay white Gaussian process with
\( G_0, G_1 \) and \( G_2 \). The average involving \( G_0 \) is \( \bar{h}_x(t-\sigma_1)x(t-\sigma_2)x(t-\sigma_3) = 0.\)
The average involving $G_1$ is:

$$
G_1 [h_1, x(t)] x(t-\sigma_1) x(t-\sigma_2) x(t-\sigma_3) \\
= \int_{-\infty}^{\infty} h_1(\tau_1) x(t-\tau_1) x(t-\tau_2) x(t-\tau_3) x(t-\sigma_1) d\tau_1 \\
= K^2 \int_{-\infty}^{\infty} h_1(\tau_1) [u(\tau_1-\sigma_1) u(\sigma_2-\sigma_3) + u(\tau_1-\sigma_2) u(\sigma_1-\sigma_3) + u(\tau_1-\sigma_3)] \\
\times u(\sigma_1-\sigma_2) d\tau_1 \\
= K^2 [u(\sigma_2-\sigma_3) h_1(\sigma_1) + u(\sigma_1-\sigma_3) h_1(\sigma_2) + u(\sigma_1-\sigma_2) h_1(\sigma_3)],
$$

and the average involving $G_2$ is:

$$
G_2 [h_2, x(t)] x(t-\sigma_1) x(t-\sigma_2) x(t-\sigma_3) \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_2(\tau_1, \tau_2) x(t-\tau_1) x(t-\tau_2) x(t-\tau_3) x(t-\sigma_1) d\tau_1 d\tau_2 \\
- K \int_{-\infty}^{\infty} h_2(\tau_2, \tau_2) x(t-\tau_1) x(t-\tau_2) x(t-\tau_3) d\tau_2 = 0,
$$

since the mean of the product of an odd number of $x$'s is zero. Therefore our final result for eqn. (17) is:

$$
y(t) x(t-\sigma_1) x(t-\sigma_2) x(t-\sigma_3) \\
= 6K^3 h_3(\sigma_1, \sigma_2, \sigma_3) + K^2 [u(\sigma_2-\sigma_3) h_1(\sigma_1) + u(\sigma_1-\sigma_3) h_1(\sigma_2) + u(\sigma_1-\sigma_2) h_1(\sigma_3)],
$$

The first term on the right-hand side of this equation involves the third-order kernel of the non-linear system that we wish to determine. The second term on the same side of the equation gives rise to impulses when $\sigma_1 = \sigma_2$, $\sigma_1 = \sigma_3$ and $\sigma_2 = \sigma_3$. But when $\sigma_1 \neq \sigma_2$, $\sigma_1 \neq \sigma_3$ and $\sigma_2 \neq \sigma_3$, the term has zero value. Thus if we apply a white Gaussian process to the unknown non-linear system and to the three-dimensional-delay circuit and take the average of the product of their outputs for various values of the delays $\sigma_1$, $\sigma_2$, and $\sigma_3$, we can express the third-order Wiener kernel of the non-linear system in terms of the cross-correlation as:

$$
h_3(\sigma_1, \sigma_2, \sigma_3) = \frac{1}{6K^3} \frac{y(t)y_3(t)}{y(t)} 	ext{ for } \sigma_1 \neq \sigma_2, \sigma_2 \neq \sigma_3, \sigma_3 \neq \sigma_1.
$$

(For the removal of the restrictions on the values of the $\sigma$ see § 7.)

§ 6. Determination of the $n$th-order Wiener Kernel

The $n$th-order Wiener kernel can be measured in the manner shown in fig. 6. The average of the product of the output of the unknown non-linear system and the output of the $n$-dimensional-delay circuit is given by:

$$
y(t)y_n(t) = \left\{ \sum_{m=0}^{\infty} G_m [h_m, x(t)] \right\} x(t-\sigma_1) x(t-\sigma_2) \ldots x(t-\sigma_n).
$$
For \( m > n \) the cross-correlation is zero. To evaluate the cross-correlation for \( m = n \), let us write the \( n \)th-degree homogeneous functional \( x(t - \sigma_1) x(t - \sigma_2) \ldots x(t - \sigma_n) \) as the leading term of the orthogonal functional \( G_n[k_n, x(t)] \) so that

\[
G_n[k_n, x(t)] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} k_n(\tau_1, \ldots, \tau_n) x(t - \tau_1) \cdots x(t - \tau_n) d\tau_1 \cdots d\tau_n + F, \tag{29}
\]

in which \( F \) is a sum of homogeneous functionals of degrees lower than \( n \).

It is clear from eqn. (9) that \( k_n(\tau_1, \ldots, \tau_n) \) in eqn. (29) is:

\[
k_n(\tau_1, \ldots, \tau_n) = u(\tau_1 - \sigma_1) \ldots u(\tau_n - \sigma_n). \tag{30}
\]

In terms of eqn. (29), the cross-correlation for the term \( m = n \) in eqn. (28) is:

\[
\overline{G_n[h_n, x(t)] x(t - \sigma_1) \ldots x(t - \sigma_n)} = \overline{G_n[h_n, x(t)] G_n[k_n, x(t)] - F}. \tag{31}
\]

**Fig. 6**

Measurement of the \( n \)th-order Wiener kernel of a non-linear system.

Since \( G_n \) is orthogonal to all functionals of degrees lower than \( n \),

\[
\overline{G_n[h_n, x(t)] F} = 0. \tag{32}
\]

Hence

\[
\overline{G_n[h_n, x(t)] x(t - \sigma_1) \ldots x(t - \sigma_n)} = \overline{G_n[h_n, x(t)] G_n[k_n, x(t)]}. \tag{33}
\]

Formulae for the mean value of the product of functionals that are members
of sets of orthogonal functionals are known (Wiener 1958, p. 41). In the present instance, it is:

\[ G_n[h_n, x(t)]G_n[k_n, x(t)] \]

\[ = n ! K^n \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n)k_n(\tau_1, \ldots, \tau_n) d\tau_1 \ldots d\tau_n \]

\[ = n ! K^n \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n)u(\tau_1 - \sigma_1) \ldots u(\tau_n - \sigma_n) d\tau_1 \ldots d\tau_n \]

\[ = n ! K^n h_n(\sigma_1, \ldots, \sigma_n). \] (34)

Note that \( k_n \) is given by eqn. (30).

Substituting eqn. (34) in eqn. (33), we obtain:

\[ G_n[h_n, x(t)]x(t - \sigma_1) \ldots x(t - \sigma_n) = n ! K^n h_n(\sigma_1, \ldots, \sigma_n). \] (35)

Our detailed work on \( h_0, h_1, h_2 \) and \( h_3 \) is in agreement with this general result. (See eqns. (11), (16) and (23).)

We now return to eqn. (28) to consider the situation for \( m < n \). It is known that if \( m \) is even, then all of the terms in \( G_m \) are functionals of even degrees; and if \( m \) is odd, then all of the terms in \( G_m \) are functionals of odd degrees. When \( n \) is even and \( m \) is even, the highest degree functional in \( G_m \) involved in eqn. (28), for \( m < n \), is of the degree \( n - 2 \). This condition means that the average

\[ x(t - \tau_1) \ldots x(t - \tau_{n-2})x(t - \sigma_1) \ldots x(t - \sigma_n), \quad \text{for } n > 2 \] (36)

has to be taken in association with the highest degree functional in \( G_m \).

Since the mean of the product of Gaussian variables can be reduced to a sum of products of the means of the products of pairs of the variables taken in all distinct ways (Schetzen 1961) and since in eqn. (36) there are two more \( \sigma \)'s than \( \tau \)'s, the result is that eqn. (36) is an impulse whenever two or more \( \sigma \)'s are equal and is zero otherwise. This fact is illustrated by eqn. (24) in the determination of \( h_3 \). Similarly, the average of the product of the \( n \)-dimensional-delay process and the other terms in \( G_m \) for \( m < n - 2 \) is an impulse when two or more \( \sigma \)'s are equal and is zero otherwise. In other words, for \( n \) even and greater than 2, we have:

\[
\begin{bmatrix}
\frac{x(t - \sigma_1) \ldots x(t - \sigma_n)}{x(t - \tau_1)x(t - \tau_2)x(t - \sigma_1) \ldots x(t - \sigma_n)} \\
\vdots \\
\frac{x(t - \tau_1) \ldots x(t - \tau_{n-2})x(t - \sigma_1) \ldots x(t - \sigma_n)}
\end{bmatrix}
\begin{bmatrix}
0 \text{ if no two } \sigma \text{'s are equal,} \\
\text{an impulse if two or more } \sigma \text{'s are equal.}
\end{bmatrix}
\] (37)

Furthermore, when \( n \) in \( x(t - \sigma_1) \ldots x(t - \sigma_n) \) is even and \( m \) in \( G_m \) is odd the cross-correlation in eqn. (28) is zero because the mean of the product of an odd number of \( x \)'s is zero.
When \( n \) in \( x(t - \sigma_1) \ldots x(t - \sigma_n) \) is odd and \( m \) in \( G_m \) is also odd, an argument similar to that just given will lead to the conclusion that for \( n \) odd and greater than 2:

\[
\begin{align*}
\frac{x(t - \tau_1)x(t - \sigma_1) \ldots x(t - \sigma_n)}{x(t - \tau_1)x(t - \tau_2)x(t - \tau_3)x(t - \sigma_1) \ldots x(t - \sigma_n)} &= \begin{cases} 
0 \text{ if no two } \sigma \text{'s are equal} \\
\text{an impulse if two or more } \sigma \text{'s are equal.} 
\end{cases}
\end{align*}
\]

This completes the discussion of eqn. (28) for \( m > n, \ m = n, \) and \( m < n. \)

Combining the results that eqn. (28) is zero for \( m > n, \) it is given by eqn. (35) for \( m = n, \) and has the properties of eqns. (37) and (38) for \( m < n, \) we obtain:

\[
h_n(\sigma_1, \ldots, \sigma_n) = \frac{1}{n!} K_n y(t)y_n(t), \quad \text{except when two or more } \sigma \text{'s are equal.}
\]

\[
(39)
\]

\section*{§ 7. Removal of Restrictions}

To develop a procedure of measurement that will be valid for all values of the \( \sigma \)'s, let us note that the first restriction on the values of the \( \sigma \)'s is in the measurement of \( h_2 \) when \( G_0 \) of the unknown system produces an impulse with \( \sigma_1 = \sigma_2 \) indicated by (14). \( G_0 \) has already been determined when we measure \( h_2. \) Hence if we subtract \( G_0 \) from the output of the unknown system, we have:

\[
\{y(t) - G_0[h_0, x(t)]\} y_2(t) = 2K^2 h_2(\sigma_1, \sigma_2) \quad \text{for all } \sigma_1 \text{ and } \sigma_2,
\]

so that

\[
h_2(\sigma_1, \sigma_2) = \frac{1}{2K^2} \{y(t) - G_0[h_0, x(t)]\} y_2(t) \quad \text{for all } \sigma_1 \text{ and } \sigma_2.
\]

Similarly, when we measure \( h_3, \) the kernel \( h_1 \) has already been measured, so that \( G_1 \) can be formed and subtracted from \( y(t). \) Since \( G_1 \) is the only term that produces impulses in the measurement of \( h_0, \) we have:

\[
\{y(t) - G_1[h_1, x(t)]\} y_3(t) = 6K^3 h_3(\sigma_1, \sigma_2, \sigma_3) \quad \text{for all } \sigma_1, \sigma_2 \text{ and } \sigma_3.
\]

Therefore,

\[
h_3(\sigma_1, \sigma_2, \sigma_3) = \frac{1}{6K^3} \{y(t) - G_1[h_1, x(t)]\} y_3(t) \quad \text{for all } \sigma_1, \sigma_2 \text{ and } \sigma_3.
\]

In fact since \( h_0 \) and \( h_2 \) are available, \( G_0 \) and \( G_2 \) may be subtracted from \( y(t) \) with the result:

\[
h_3(\sigma_1, \sigma_2, \sigma_3) = \frac{1}{6K^3} \left\{y(t) - \sum_{m=0}^{2} G_m[h_m, x(t)]\right\} y_3(t) \quad \text{for all } \sigma_1, \sigma_2 \text{ and } \sigma_3.
\]

\[
(44)
\]
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For the measurement of $h_n$, instead of (39), we would have the unrestricted expression:

$$h_n(\sigma_1, \ldots, \sigma_n) = \frac{1}{n! K^n} \left\{ y(t) - \sum_{m=0}^{n-1} G_n[h_m, x(t)] \right\} y_n(t); \text{ for all } \sigma_1, \ldots, \sigma_n.$$  \hspace{1cm} (45)

§ 8. Experiment

In an experiment carried out by Widnall (1962), this theory was applied to the experimental determination of the Wiener kernels of the non-linear system shown in fig. 7. As indicated in the figure, the linear system has the impulse response $k(t)$, input $x(t)$, and output $v(t)$. The square-law device has the characteristic:

$$y(t) = ax^2(t),$$  \hspace{1cm} (46)

so that

$$y(t) = a \left[ \int_{-\infty}^{\infty} k(\tau)x(t-\tau) \, d\tau \right]^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a k(\tau_1) k(\tau_2) x(t-\tau_1) x(t-\tau_2) \, d\tau_1 \, d\tau_2.$$  \hspace{1cm} (47)

This functional relationship for $y(t)$ can be expressed in terms of the $G$-functionals as:

$$y(t) = G_0[h_0, x(t)] + G_2[h_2, x(t)],$$  \hspace{1cm} (48)

in which

$$G_0[h_0, x(t)] = y(t) = K \int_{-\infty}^{\infty} a k^2(\tau_1) \, d\tau_1$$  \hspace{1cm} (49)

and

$$G_2[h_2, x(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a k(\tau_1) k(\tau_2) x(t-\tau_1) x(t-\tau_2) \, d\tau_1 \, d\tau_2$$

$$- K \int_{-\infty}^{\infty} a k^2(\tau_1) \, d\tau_1,$$  \hspace{1cm} (50)

so that the second-order Wiener kernel is:

$$h_2(\tau_1, \tau_2) = a k(\tau_1) k(\tau_2).$$  \hspace{1cm} (51)

Fig. 7

The non-linear system whose second-order Wiener kernel was measured.

The impulse response, $k(t)$, of the particular linear system, was that of an RLC network. A block diagram of the experiment is shown in fig. 8. As shown, the input of the non-linear system, $x(t)$, is the output of a Gaussian noise generator whose spectrum is flat over the band of frequencies of interest. This is the practical realization of a white Gaussian noise source. The data needed for the cross-correlation were recorded automatically by a
Y. W. Lee and M. Schetzen on the dual-channel analogue-to-digital conversion system. Each measured point of the kernel is the result of averaging 30,000 sample products by the digital computer. The experimental results are displayed in the form of a three-dimensional model shown in fig. 9. The r.m.s. error was 0.6% of the maximum value of the kernel, and the calculation for 20 x 20 points required 15 min. of IBM 7090 computer time.

Fig. 8

Block diagram of the experiment.

Fig. 9

Three-dimensional model of the measured second-order Wiener kernel, $h_2(\tau_1, \tau_2)$. 
§ 9. Discussion

In comparison with the Wiener method of measurement of the kernels, the present method has the advantage of great simplicity. Digital computation and tape recording are particularly helpful in the application of the method. As we see from the theory of the method, the only necessary data for the characterization of a non-linear system—that is, the determination of its kernels of all orders—are the record of the white Gaussian process that is fed into the non-linear system and the corresponding output of the system. The record can be in the form of a dual-track recording on magnetic tape.

In the Wiener method of measurement the basis is the orthogonal expansion of the kernels and the representation of the orthogonal sets of functions by a system of linear networks and a system of non-linear no-memory networks. Since, in practical application, the number of terms in an expansion must be finite, the Wiener method involves an error that is attributable to the truncation of the expansion. We know, however, that the error in the representation of a function by a finite orthogonal set of functions is the minimum integral-square error. On the other hand, the method discussed in this paper does not depend upon a series expansion of the kernels in any form. Hence another advantage of the method is that it involves no approximation error. In both methods, as we are aware, there is, among other errors, an error that is the result of using a finite time in taking the necessary average values.

We also note that the present method is a point-by-point method, whereas the Wiener method is a minimum-integral-square-error approximation method over the entire range of time. The determination of a set of coefficients determines the approximation over the entire range of time. We see that under certain circumstances these methods may complement each other. For instance, the Wiener method may indicate quickly the parts of the kernel curve that need greater details. These details may be more effectively obtained by the present method. Again, in expanding the kernels by the Wiener method, we may wish to know whether the approximation is sufficiently good. A comparison of the approximation with the measurement by the present method should be a good check.

§ 10. Measurement with a Non-white Gaussian Input

The theory that we have presented can be generalized to the case for a non-white Gaussian input process.

Consider that the system $N$ shown in fig. 10 is to be characterized with an input, $z(t)$, which is a non-white Gaussian process for which the power

![Fig. 10](image-url)

Non-linear system with non-white Gaussian noise input.
density spectrum, $\Phi_{zz}(\omega)$, is factorable (Lee 1960). It then can be written:

$$\Phi_{zz}(\omega) = \Phi_{zz}^+(\omega) \Phi_{zz}^-(\omega),$$

(52)
in which $\Phi_{zz}^+(\omega)$ is the complex conjugate of $\Phi_{zz}^-(\omega)$; also all of the poles and zeros of $\Phi_{zz}^+(\omega)$ are in the left half of the complex $s$-plane in which $s = \sigma + j\omega$. Thus $\Phi_{zz}^+(\omega)$ and $1/\Phi_{zz}^+(\omega)$ are each realizable as the transfer function of a linear system. We can then consider the system of fig. 10 in the equivalent form shown in fig. 11, in which the transfer functions of the two linear systems, $k_1(t)$ and $k_2(t)$, are:

$$K_1(\omega) = \frac{1}{\Phi_{zz}^+(\omega)},$$

$$K_2(\omega) = \Phi_{zz}^+(\omega).$$

(53)

Also, as shown, the system $A$ is the system formed by the tandem connection of the linear system $k_2(t)$ and the system $N$. We observe that $x(t)$, the input to the system $A$, is a Gaussian white process whose power density spectrum is 1 watt per radian per second. Thus, according to eqn. (39), the Wiener kernels, $h_n$, of the system $A$ are:

$$h_n(\tau_1, \ldots, \tau_n) = \frac{(2\pi)^n}{\sqrt{n!}} y(t) x(t - \tau_1) \ldots x(t - \tau_n),$$

(54)

except when two or more $\tau$'s are equal.

We therefore need to know the cross-correlation function

$$\phi_{yx}(\tau_1, \ldots, \tau_n) = y(t) x(t - \tau_1) \ldots x(t - \tau_n)$$

(55)
in order to determine the kernels, $h_n$, of the system $A$. Since only $z(t)$ is available to us, we shall express the desired correlation function in terms of the cross-correlation between the output, $y(t)$, and a multi-dimensional delay of the input, $z(t)$. By substituting the relation

$$x(t) = \int_0^\infty k_1(\sigma) z(t - \sigma) \, d\sigma$$

(56)
in eqn. (55), the desired correlation function can be expressed as:

$$\phi_{yz}(\tau_1, \ldots, \tau_n) = \int_0^\infty k_1(\sigma_1) d\sigma_1 \ldots \int_0^\infty k_1(\sigma_n) d\sigma_n \phi_{yx}(\tau_1 - \sigma_1, \ldots, \tau_n - \sigma_n),$$

(57)
in which

$$\phi_{yx}(\tau_1, \ldots, \tau_n) = y(t) z(t + \tau_1) \ldots z(t + \tau_n)$$

(58)
Wiener Kernels of a Non-linear System by Cross-correlation

is the cross-correlation between the output, \( y(t) \), and a multi-dimensional delay of the input, \( z(t) \). In the frequency domain, eqn. (57) can be expressed as:

\[
\Phi_{yz}(\omega_1, \ldots, \omega_n) = K_1(\omega_1)K_2(\omega_2) \cdots K_n(\omega_n)\Phi_{zz}(\omega_1, \ldots, \omega_n),
\]

(59)
in which

\[
\Phi_{yz}(\omega_1, \ldots, \omega_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \exp(-j\omega_1 \tau_1) d\tau_1 \cdots \int_{-\infty}^{\infty} \exp(-j\omega_n \tau_n) \times d\tau_n \Phi_{zz}(\tau_1, \ldots, \tau_n)
\]

(60)
and the transfer function \( K_1(\omega) \) is given by:

\[
K_1(\omega) = \int_{-\infty}^{\infty} k_1(t) \exp(-j\omega t) dt.
\]

(61)
Substituting eqn. (53) in eqn. (59), we have the desired relation in the frequency domain:

\[
\Phi_{yz}(\omega_1, \ldots, \omega_n) = \frac{\Phi_{yz}(\omega_1, \ldots, \omega_n)}{\Phi_{zz}^{-1}(\omega_1) \cdots \Phi_{zz}^{-1}(\omega_n)}.
\]

(62)
Either eqn. (57) or eqn. (62) can be used to determine the kernels, \( h_n \), as given by eqn. (54) in terms of the measured cross-correlation function between the output and a multi-dimensional delay of the input, \( z(t) \).

![Fig. 12](image)

**Representation of the expansion for system N.**

Once the kernels \( h_n \) have been determined, a representation of the system N is as given in fig. 12 (a) which can be redrawn as in fig. 12 (b). The outputs of the parallel branches in fig. 12 (b) are seen to be orthogonal for the input \( z(t) \). Thus, we have expanded the non-linear system N in a set of functionals that are orthogonal for Gaussian inputs with a power density spectrum of \( \Phi_{zz}(\omega) \). Note that for this procedure, we never need construct CON.
On the Wiener Kernels of a Non-linear System by Cross-correlation

either of the linear systems \( k_1(t) \) or \( k_2(t) \). The functionals that are orthogonal for the input \( z(t) \) can be calculated by substituting eqn. (56) in eqn. (2). We shall call these functionals \( L_n \). The first few functionals are:

\[
\begin{align*}
L_0[h_0, z(t)] &= h_0, \\
L_1[h_1, z(t)] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\sigma_1)k_1(\tau_1 - \sigma_1)z(t - \tau_1) d\sigma_1 d\tau_1, \\
L_2[h_2, z(t)] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_2(\sigma_1, \sigma_2)k_1(\tau_1 - \sigma_1)k_1(\tau_2 - \sigma_2) \\
&\quad \times z(t - \tau_1)z(t - \tau_2) d\sigma_1 d\sigma_2 d\tau_1 d\tau_2 - 2\pi \int_{-\infty}^{\infty} h_2(\sigma_2, \sigma_3) d\sigma_2 \\
L_3[h_3, z(t)] &= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} h_3(\sigma_1, \sigma_2, \sigma_3)k_1(\tau_1 - \sigma_1)k_1(\tau_2 - \sigma_2) \\
&\quad \times k_1(\tau_3 - \sigma_3)z(t - \tau_1)z(t - \tau_2)z(t - \tau_3) d\sigma_1 d\sigma_2 d\sigma_3 d\tau_1 d\tau_2 d\tau_3 \\
&\quad - 3(2\pi) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_3(\sigma_1, \sigma_2, \sigma_3)k_1(\tau_1 - \sigma_1) \\
&\quad \times z(t - \tau_1) d\sigma_1 d\sigma_2 d\tau_1, \quad (63)
\end{align*}
\]

in which

\[
k_1(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(j\omega t)}{\Phi_{zz}(\omega)} d\omega. \quad (64)
\]

In terms of these functionals, the output of the system \( N \) for an input \( z(t) \) can be represented by the orthogonal expansion:

\[
y(t) = \sum_{n=0}^{\infty} L_n[h_n, z(t)], \quad (65)
\]

which is equivalent to eqn. (1) for the case in which the input is a white Gaussian process.

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